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Effects of Foldings on Free Product of Fundamental Groups

M.El-Ghoul, A. E.El-Ahmady, H.Rafat and M.Abu-Saleem

(Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt)

E-mail: hishamrafat2005@yahoo.com

Abstract: In this paper, we will introduce free fundamental groups of some types of manifolds. Some types of conditional foldings restricted on the elements on free group and their fundamental groups are deduced. Also, the fundamental group of the limit of foldings on a wedge sum of two manifolds is obtained. Theorems governing these relations will be achieved.

Key Words: Manifolds, Folding, fundamental group, Free group

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§1. Introduction

In this article the concept of foldings will be discussed from viewpoint of algebra. The effect of foldings on the manifold M or on a finite number of product manifolds $M_1xM_2x...xM_n$ on the fundamental group $\pi_1(M)$ and $\pi_1(M_1xM_2x\cdots xM_n)$ will be presented. The folding of a manifold was, firstly introduced by Robertson 1977 [14]. More studies on the folding of many types of manifolds were studied in [2-4 and 6-9]. The unfolding of a manifold introduced in [5]. Some application of the folding of a manifold discussed in [1]. The fundamental groups of some types of a manifold are discussed in [10-13].

§2. Definitions

1. The set of homotopy classes of loops based at the point $x_\circ$ with the product operation $[f][g]=[f\cdot g]$ is called the fundamental group and denoted by $\pi_1(X,x_\circ)$ [11].

2. Let M and N be two smooth manifolds of dimension m and n respectively. A map $f:M\rightarrow N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma:I\rightarrow M$ the induced path $f\circ\gamma:I\rightarrow N$ is piecewise geodesic and of the same length as γ [14]. If f does not preserve length it is called topological folding [9].

3. Let M and N be two smooth manifolds of the same dimension. A map $g:M\rightarrow N$ is said to be unfolding of M into N if every piecewise geodesic path $\gamma:I\rightarrow M$, the induced path $g\circ\gamma:I\rightarrow N$ is piecewise geodesic with length greater than γ [5].

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4. Given spaces X and Y with chosen points $x_o \in X$ and $y_o \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained identifying x_o and y_o to a single point [11].

5. Let S be an arbitrary set. A free group on the set S is a group F together with a function $\phi : S \rightarrow F$ such that the following condition holds: For any function $\psi : S \rightarrow H$, there exist a unique homomorphism $f : F \rightarrow H$ such that $f \circ \phi = \psi$ [12].

§3. Main Results

Paving the stage to this paper, we then introduce the following

(1) $\pi_1(T) = \{([\alpha_1]^k, [\beta_1]^m), ([\alpha_2]^k, [\beta_2]^m), ..., ([\alpha_n]^k, [\beta_n]^m); [\alpha_i], [\beta_i] \in \pi_1(S^1), k, m \in Z, k \neq 0, m \neq 0, i = 1, 2, ..., n\}$

(2) $\pi_1(T)\text{mod}(k, m) = \{([\alpha_1], [\beta_1]), ([\alpha_2], [\beta_2]), ..., ([\alpha_n], [\beta_n]) : [\alpha_i]^k = 1, [\beta_i]^m = 1\ [\alpha_i], [\beta_i] \in \pi_1(S^1), k, m \in Z^+, k \neq 0, m \neq 0, i = 1, 2, ..., n\}$.

Where, $\pi_1(S^1)$ is a fundamental group of the circle ,T is the torus $[\alpha]^n = \underbrace{[\alpha] \times [\alpha] \times .... \times [\alpha]}_{n-terms}$, and $T^n = \underbrace{T \times T \times .... \times T}_{n-terms}$.

Let $\pi_1(S_1^1)$, $\pi_1(S_2^1)$ be two fundamental groups. Then the free product of $\pi_1(S_1^1)$, $\pi_1(S_2^1)$ is the group $\pi_1(S_1^1) * \pi_1(S_2^1)$consisting of all reduced words $a_1a_2a_3....a_m$ of an arbitrary finite length $m \geq 0$ such that $a_i \in \pi_1(S_1^1)$ or $a_i \in \pi_1(S_2^1), i = 1, 2, ..., m$, then we can represent the elements a_i as of the forms $a_i = [\alpha]^{n_i}$ or $a_i = [\beta]^{n_i}$ where $n_i \in Z, n_i \neq 0$ and α, β are two loops that goes once a round S_1^1, S_2^1 respectively. Also, if $F : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ is a folding, then the induced folding $\overline{F}$:$\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$ has the following forms:

$$\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = \overline{F}(\pi_1(S_1^1)) * \pi_1(S_2^1),$$
$$\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = \pi_1(S_1^1) * \overline{F}(\pi_1(S_2^1)),$$
$$\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = \overline{F}(\pi_1(S_1^1)) * \overline{F}(\pi_1(S_2^1)).$$

Theorem 3.1 *If $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$, $i = 1, 2$ are two types of of foldings, where $F_I(S_j^1) =.S_j^1$, $j = 1, 2$, then there are induced foldings $\overline{F_i}$:$\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$ such that $\overline{F_i}(\pi_1(S_1^1)) * \pi_1(S_2^1) \approx Z$.*

Proof First, let $F_1 : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ is folding such that $F_1(S_1^1) = S_1^1$, $F_1(S_2^1) = S_1^1$ as in Fig.1. Then we can express each element $g = a_1a_2a_3....a_m$, $m \geq 1$ of $\pi_1(S_1^1) * \pi_1(S_2^1)$ in the following forms

$[\alpha]^{n_1} [\beta]^{n_2} [\alpha]^{n_3} .... [\alpha]^{n_{m-1}} [\beta]^{n_m}$, $[\alpha]^{n_1} [\beta]^{n_2} [\alpha]^{n_3} \cdots [\beta]^{n_{m-1}} [\alpha]^{n_m}$,

$[\beta]^{n_1} [\alpha]^{n_2} [\beta]^{n_3} .... [\beta]^{n_{m-1}} [\alpha]^{n_m}$, or $[\beta]^{n_1} [\alpha]^{n_2} [\beta]^{n_3} .... [\alpha]^{n_{m-1}} [\beta]^{n_m}$,

where $n_1, n_2, \cdots, n_m$ are nonzero integers and $[\alpha]^{n_k} \in \pi_1(S_1^1), [\beta]^{n_k} \in \pi_1(S_2^1), k = 1, 2, ..m$.

Then, the induced folding of the element g is

$$\begin{aligned}
&\overline{F_1}(g) = \overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m}),\\
&\overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m}),\\
&\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m}),\\
&\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})....\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m}).
\end{aligned}$$

Since $\overline{F_1}([\alpha]^{n_k}) = [\alpha]^{n_k}$, $\overline{F_1}([\beta]^{n_k}) = [\alpha]^{n_k}$ it follows that $\overline{F_1}(a_1a_2a_3...a) = [\alpha]^{(n_1+n_2+\cdots+n_m)}$. Hence, there is an induced folding $\overline{F_i}:\pi_1(S_1^1)*\pi_1(S_2^1) \longrightarrow \pi_1(S_1^1)*\pi_1(S_2^1)$ such that $\overline{F_i}(\pi_1(S_1^1)*\pi_1(S_2^1)) = \pi_1(S_1^1)$, and so $\overline{F_i}(\pi_1(S_1^1)*\pi_1(S_2^1)) \approx Z$. Similarly , if $F_2 : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ is folding, such that $F_2(S_1^1) = S_2^1$, $F_2(S_2^1) = S_2^1$, then there is an induced folding $\overline{F_2}:\pi_1(S_1^1)*\pi_1(S_2^1) \longrightarrow \pi_1(S_1^1)*\pi_1(S_2^1)$ such that $\overline{F_2}(\pi_1(S_1^1)*\pi_1(S_2^1)) \approx Z$.

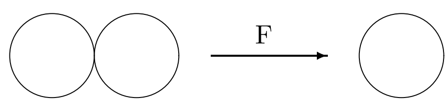

Fig.1

Theorem 3.2 *If $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1, i = 1,2$ are two types of foldings such that $F_i(S_j^1) = S_i^1, j = 1,2$. Then,$\pi_1(\lim\limits_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ is isomorphic to Z.*

Proof Let $F_i(S_j^1) = S_i^1$ then $\lim\limits_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1) = S_i^1$ as in Fig.2. Thus, $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ $=S_i^1$, Therefore $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ is isomorphic to Z. □

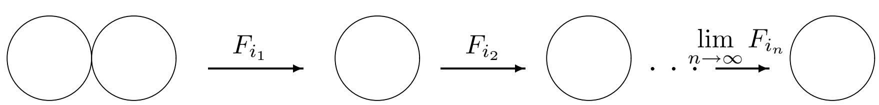

Fig.2

Theorem 3.3 *Let $F : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ be a folding, where $F(S_i^1) \neq S_i^1, i = 1,2.$Then there is an induced folding $\overline{F} : \pi_1(S_1^1)*\pi_1(S_2^1) \longrightarrow \pi_1(S_1^1)*\pi_1(S_2^1)$such that $\overline{F}\pi_1(S_1^1)*\pi_1(S_2^1)) = 0$.*

Proof Let $F : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ be a folding such that $F(S_1^1) \neq S_1^1, F(S_i^1) \neq S_i^1$ as in Fig. (3) .Then, we can express each element $g = a_1a_2a_3....a_m$, $m \geq 1$ of $\pi_1(S_1^1)*\pi_1(S_2^1)$ in the following forms:

$$\begin{aligned}
&[\alpha]^{n_1}[\beta]^{n_2}[\alpha]^{n_3}\cdots[\alpha]^{n_{m-1}}[\beta]^{n_m}, \quad [\alpha]^{n_1}[\beta]^{n_2}[\alpha]^{n_3}\cdots[\beta]^{n_{m-1}}[\alpha]^{n_m},\\
&[\beta]^{n_1}[\alpha]^{n_2}[\beta]^{n_3}\cdots[\beta]^{n_{m-1}}[\alpha]^{n_m}, \quad [\beta]^{n_1}[\alpha]^{n_2}[\beta]^{n_3}\cdots[\alpha]^{n_{m-1}}[\beta]^{n_m},
\end{aligned}$$

where $n_1, n_2, \cdots, n_m$ are nonzero integers and $[\alpha]^{n_k} \in \pi_1(S_1^1), [\beta]^{n_k} \in \pi_1(S_2^1), k = 1,2,\cdots,m$.

Then the induced folding of the element g is

$$\begin{aligned}
&\overline{F_1}(g) = \overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m})\\
&= [\alpha]^{n_1}\,[\beta]^{n_2}\,[\alpha]^{n_3}\cdots[\alpha]^{n_{m-1}}\,[\beta]^{n_m}\,,\\
&\overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m})\\
&= [\alpha]^{n_1}\,[\beta]^{n_2}\,[\alpha]^{n_3}\cdots[\beta]^{n_{m-1}}\,[\alpha]^{n_m}\,,\\
&\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m})\\
&= [\beta]^{n_1}\,[\alpha]^{n_2}\,[\beta]^{n_3}\cdots[\beta]^{n_{m-1}}\,[\alpha]^{n_m}\,,\\
&\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})\cdots\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m})\\
&= [\beta]^{n_1}\,[\alpha]^{n_2}\,[\beta]^{n_3}\cdots[\alpha]^{n_{m-1}}\,[\beta]^{n_m}\,.
\end{aligned}$$

It follows from $\left[\overset{\curlywedge}{\alpha}\right],\left[\overset{\curlywedge}{\beta}\right] \longrightarrow$ identity element, that there is an induced folding $\overline{F}:\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$such that $\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = 0$.

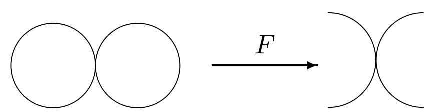

Fig.3

Corollary 1 *If $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1, i = 1, 2$ are two types of foldings such that*

$$F_i(S_i^1) = S_i^1, F_j(S_i^1) \neq S_i^1, j = 1, 2, i \neq j.$$

Then there are induced foldings $\overline{F_i}:\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$such that $\overline{F_i}(\pi_1(S_1^1) * \pi_1(S_2^1)) \approx Z$.

Theorem 4 *If $F : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ is a folding such that $F(S_i^1) \neq S_i^1, i = 1, 2$. Then,*

$$\pi_1(\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1))$$

is the identity group.

Proof If $F(S_i^1) \neq S_i^1$, $i = 1, 2$ then $\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1)$ is a point as in Fig.4, and so $\pi_1(\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1))$ is the fundamental group of a point. Therefore, we get that $\pi_1(\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1)) = 0$. □

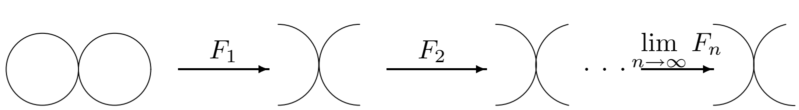

Fig.4

Theorem 5 *If $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1, i = 1, 2$ are two types of foldings such that $F_i(S_i^1) = S_i^1, F_j(S_i^1) \neq S_i^1, j = 1, 2, i \neq j$. Then $\pi_1(\lim_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ is isomorphic to Z.*

Proof It follows from $F_i(S_i^1) = S_i^1, F_j(S_i^1) \neq S_i^1, j = 1, 2, i \neq j$.that the limit of one circle is a circle and the limit of the other circle is a point, so $\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1)) = S_i^1$ as in Fig.5. Thus, $\pi_1(\lim_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1)) = \pi_1(S_i^1)$. Therefore $\pi_1(\lim_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ is isomorphic to Z. $\square$

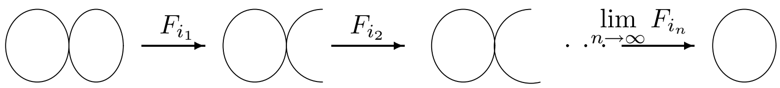

Fig.5

Now, we will generalize the above concepts for the tours Consider $\pi_1(T_1^1)$,$\pi_1(T_2^1)$, are two fundamental groups. Then, the free product of $\pi_1(T_1^1)$,$\pi_1(T_2^1)$, is the group $\pi_1(T_1^1) * \pi_1(T_2^1)$ consisting of all reduced words of $a_1a_2a_3....a_m$ of an arbitrary finite length $m \geq 0$ such that

$a_i \in \pi_1(T_1^1)$ or $a_i \in \pi_1(T_2^1)$ and so, we can represent the elements a_i as of the forms $a_i = ([\alpha_1]^{n_i}, [\beta_1]^{k_i})$ or $a_i = ([\alpha_2]^{n_i}, [\beta_2]^{k_i})$ where $n_i, k_i \in Z, n_i \neq 0$, $k_i \neq 0$ where $([\alpha_1]^{n_i}, [\beta_1]^{k_i}) \in \pi_1(T_1^1)$, $([\alpha_2]^{n_i}, [\beta_2]^{k_i}) \in \pi_1(T_2^1)$and α_j, β_j are loops that goes once a round the generators of T_j for $j = 1, 2$.Then if $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$is a folding, then the induced folding $\overline{F}:\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ has the following forms:

$$\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \overline{F}(\pi_1(T_1^1)) * \pi_1(T_2^1),$$
$$\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \pi_1(T_1^1) * \overline{F}(\pi_1(T_2^1)),$$
$$\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \overline{F}(\pi_1(T_1^1)) * \overline{F}(\pi_1(T_2^1)).$$

Theorem 6 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1, 2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1, 2$. Then, there are induced foldings $\overline{F_i}:\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ such that $\overline{F_i}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z \times Z$.*

Proof First, if $F_1 : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ is a folding such that $F_1(T_1^1) = T_1, F_1(T_2^1) = T_1$ as in Fig.6. Then we can express each element $g = a_1a_2...a_m, m \geq 1$ of $\pi_1(T_1^1) * \pi_1(T_2^1)$ in the following forms.

$$([\alpha_1]^{n_1}, [\beta_1]^{k_1})([\alpha_2]^{n_2}, [\beta_2]^{k_2})([\alpha_1]^{n_3}, [\beta_1]^{k_3}) \cdots ([\alpha_1]^{n_{m-1}}, [\beta_1]^{k_{m-1}})([\alpha_2]^{n_m} [\beta_2]^{k_m}),$$
$$([\alpha_1]^{n_1}, [\beta_1]^{k_1})([\alpha_2]^{n_2}, [\beta_2]^{k_2})([\alpha_1]^{n_3}, [\beta_1]^{k_3}) \cdots ([\alpha_2]^{n_{m-1}}, [\beta_2]^{k_{m-1}})([\alpha_1]^{n_m} [\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1}, [\beta_2]^{k_1})([\alpha_1]^{n_2}, [\beta_1]^{k_2})([\alpha_2]^{n_3}, [\beta_2]^{k_3}) \cdots ([\alpha_2]^{n_{m-1}}, [\beta_2]^{k_{m-1}})([\alpha_1]^{n_m} [\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1}, [\beta_2]^{k_1})([\alpha_1]^{n_2}, [\beta_1]^{k_2})([\alpha_2]^{n_3}, [\beta_2]^{k_3}) \cdots ([\alpha_1]^{n_{m-1}}, [\beta_1]^{k_{m-1}})([\alpha_1]^{n_m} [\beta_1]^{k_m}),$$

where $n_1, n_2, \cdots, n_m$, $k_1, k_2, \cdots, k_m$ are nonzero integers,

$$([\alpha_i]^{n_1}, [\beta_i]^{k_1}) \in \pi_1(T_1^1), ([\alpha_i]^{n_2}, [\beta_i]^{k_2}) \in \pi_1(T_2^1).$$

Since $\overline{F_1}([\alpha_1]^{n_1}, [\beta_1]^{k_1}) = ([\alpha_1]^{n_1}, [\beta_1]^{k_1}), \overline{F_1}([\alpha_2]^{n_1}, [\beta_2]^{k_1}) = ([\alpha_1]^{n_1}, [\beta_1]^{k_1})$, it follows that there is an induced folding $\overline{F_i}:\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ such that $\overline{F_1}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \pi_1(T_1^1)$, and so $\overline{F_1}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z \times Z$. Similarly , if $F_2 : T_1^1 \vee T_2^1 \longrightarrow$

$T_1^1 \vee T_2^1$ is folding, such that $F_2(T_1^1) = T_1, F_2(T_2^1) = T_1$, then there is an induced folding $\overline{F_2}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \pi_1(T_1^1)$ such that $\overline{F_2}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z \times Z$. □

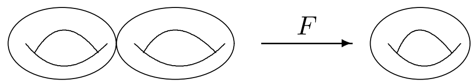

Fig.6

Theorem 7 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1,2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1,2$. Then $\pi_1(\lim_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1)) \approx Z \times Z$.*

Proof If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1,2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1,2$, then $\lim_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1) = T_i^1$ as in Fig.7. Thus $\pi_1(\lim_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1)) = \pi_1(T_i^1)$,since $\pi_1(T_i^1) \approx Z \times Z$ we have $\pi_1(\lim_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1)) \approx Z \times Z$. □

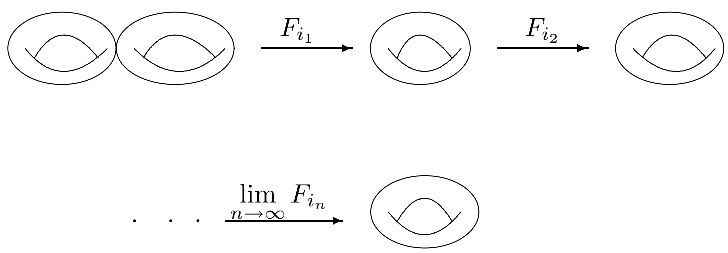

Fig.7

Corollary 2 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1,2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1,2$. Then $\pi_1(\lim_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1))$ is a free Abelian group of rank $2n$.*

Proof Since $F_i(T_j^1) = T_i, j = 1,2$ we have the following chain $T_1^1 \vee T_2^1 \xrightarrow{F_{i_1}} T_i^n \xrightarrow{F_{i_2}} T_i^n .... \xrightarrow{\lim_{n\to\infty} F_{i_n}} T_i^n$Since $\pi_1(T_i^n) = \underbrace{\pi_1(T_i \times T_i \times .... \times T_i)}_{n-terms}$, , it follows that $\pi_1(T_i^n) \approx \underbrace{Z \times Z \times .... \times Z}_{2n-terms}$. Hence ,$\pi_1(\lim_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1))$ is a free Abelian of rank $2n$. □

Theorem 8 *If $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ is a folding by cut such that $F_1(T_1^1) \neq T_1, F_1(T_2^1) \neq T_1$ .Then there is induced folding $\overline{F}$:$\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ such that $\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z * Z$.*

Proof Let $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ is a folding such that $F_1(T_1^1) \neq T_1, F_1(T_2^1) \neq T_1$as in Fig.8. Then, we can express each element $g = a_1 a_2 \cdots a_m, m \geq 1$ of $\pi_1(T_1^1) * \pi_1(T_2^1)$ in the following forms

$$([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_2]^{n_m}[\beta_2]^{k_m}),$$
$$([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})\cdots([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),$$

where $n_1, n_2, \cdots, n_m$, $k_1, k_2, \cdots, k_m$ are nonzero integers and

$$([\alpha_i]^{n_1},[\beta_i]^{k_1}) \in \pi_1(T_1^1), ([\alpha_i]^{n_2},[\beta_i]^{k_2}) \in \pi_1(T_2^1).$$

Then, the induced folding of the element g is

$$\begin{aligned}
\overline{F}(g) &= \overline{F}([\alpha_1]^{n_1},[\beta_1]^{k_1})\overline{F}([\alpha_2]^{n_2},[\beta_2]^{k_2})\overline{F}([\alpha_1]^{n_3},[\beta_1]^{k_3})\\
&\quad \cdots\overline{F}([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})\overline{F}([\alpha_2]^{n_m}[\beta_2]^{k_m})\\
&= ([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_2]^{n_m}[\beta_2]^{k_m}),\\
&\quad \overline{F}([\alpha_1]^{n_1},[\beta_1]^{k_1})\overline{F}([\alpha_2]^{n_2},[\beta_2]^{k_2})\overline{F}([\alpha_1]^{n_3},[\beta_1]^{k_3})\\
&\quad \cdots\overline{F}([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})\overline{F}([\alpha_1]^{n_m}[\beta_1]^{k_m})\\
&= ([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),\\
&\quad \overline{F}([\alpha_2]^{n_1},[\beta_2]^{k_1})\overline{F}([\alpha_1]^{n_2},[\beta_1]^{k_2})\overline{F}([\alpha_2]^{n_3},[\beta_2]^{k_3})\\
&\quad \cdots\overline{F}([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})\overline{F}([\alpha_1]^{n_m}[\beta_1]^{k_m})\\
&= ([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),\\
&\quad \overline{F}([\alpha_2]^{n_1},[\beta_2]^{k_1})\overline{F}([\alpha_1]^{n_2},[\beta_1]^{k_2})\overline{F}([\alpha_2]^{n_3},[\beta_2]^{k_3})\\
&\quad \cdots\overline{F}([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})\overline{F}([\alpha_1]^{n_m}[\beta_1]^{k_m})\\
&= ([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})....([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}).
\end{aligned}$$

It follows from $[\widehat{\beta}_1], [\widehat{\beta}_2] \to 0$ (identity element) that there is an induced folding such that $\overline{F}:\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$. Therefore, $\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z * Z$. □

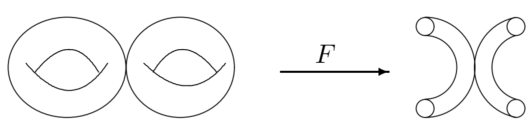

Fig.8

Corollary 3 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1, 2$ are two types of foldings such that $F_i(T_j^1) = T_i^1, F_j(T_i^1) \neq T_i^1, i, j = 1, 2, i \neq j$. Then there are induced foldings $\overline{F_i}:\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$. such that $\overline{F_i}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx (Z \times Z) * Z$.*

Theorem 9 *If $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ are a folding by cut such that $F(T_i^1) \neq T_i^1$, $for\ i = 1, 2$. Then $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1))$, is a free group of rank ≤ 2 or identity group.*

Proof Consider ,$F(T_i^1) \neq T_i^1$, $for\ i = 1, 2$, then we have the following: $\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1) = S_1^1 \vee S_2^1$ as in Fig.9(a) then ,$\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)) \approx \pi_1(S_1^1) \vee \pi_1(S_2^1)$, and so $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1))$

$\approx Z * Z$. Hence, $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is a free group of rank 2.Also, If $\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)$ as in Fig.9(b), then $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)) = 0$. Moreover, if $\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)$ as in Fig.9(c), then $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)) \approx \pi_1(S_1^1) \approx Z$ .Therefore, $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is a free group of rank ≤ 2 or identity group. □

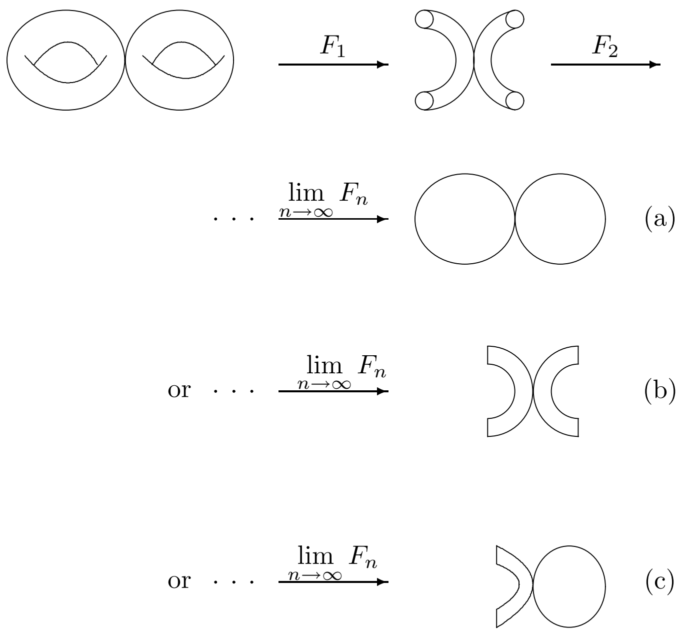

Fig.9

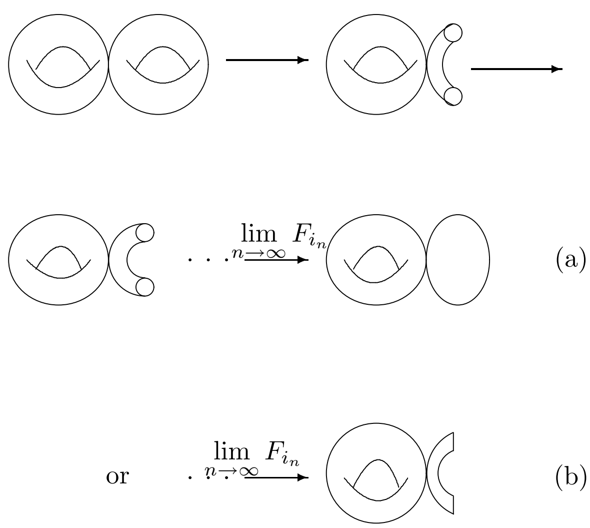

Fig.10

Theorem 10 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$,$i = 1,2$ are two types of foldings such that*

$F_i(T_i^1) = T_i^1, F_j(T_i^1) \neq T_i^1, i, j = 1, 2, i \neq j$. Then $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is either isomorphic $(Z \times Z) * Z$ to or $(Z \times Z)$.

Proof Since $F_i(T_i^1) = T_i^1, F_j(T_i^1) \neq T_i^1, i, j = 1, 2, i \neq j$, we have the following:

If $\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1) = T_i^1 \vee S_i^1$ as in Fig.10(a), then $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1)) = \pi_1(T_i^1 \vee S_i^1) \approx (Z \times Z) * Z$. Also, if $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1)) = \pi_1(T_i^1)$ as in Fig.10(b) then $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1)\pi_1(T_i^1) \approx Z \times Z$. Hence, $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is either isomorphic to $(Z \times Z) * Z$ or

$(Z \times Z)$. □

Theorem 11 *If $F : T_1^n \vee T_2^n \longrightarrow T_1^n \vee T_2^n$ is a folding such that $F(T_1^n) = T_1^n$ and $F(T_2^n) \neq T_2^n$ where $F(T_2^n) = \underbrace{F(T_2^1) \times F(T_2^1) \times .... \times F(T_2^1)}_{n-terms}$,$F(T_2^1) \neq T_2^1$ is a folding by cut. Then, $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^n \vee T_2^n))$ is isomorphic to $\underbrace{(Z \times Z \times .... \times Z)}_{2n-terms} * \underbrace{Z \times Z \times .... \times Z}_{n-terms}$.*

Proof Since $F(T_1^n) = T_1^n$, $F(T_2^n) \neq T_2^n$ we have the following chain:

$$T_1^n \vee T_2^n \underset{\rightarrow}{F} T_1^n \vee \underbrace{F(S_1^1) \times S_2^1 \times F(S_1^1) \times S_2^1 \times \cdots \times F(S_1^1) \times S_2^1}_{2n-terms} \underset{\rightarrow}{F},$$

$$T_1^n \vee T_2^n \underset{\rightarrow}{F} T_1^n \vee \underbrace{F(S_1^1) \times S_2^1 \times F(S_1^1) \times S_2^1 \times \cdots \times F(S_1^1) \times S_2^1}_{2n-terms} \underset{\rightarrow}{F},$$

$$T_1^n \vee \underbrace{F(F(S_1^1)) \times S_2^1 \times F(F(S_1^1)) \times S_2^1 \times \cdots \times F(F(S_1^1)) \times S_2^1}_{2n-terms} \xrightarrow{\lim\limits_{n\to\infty} F_n},$$

$$T_1^n \vee \underbrace{(S_2^1 \times S_2^1 \times \cdots \times S_2^1)}_{n-terms}.$$

Hence, $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^n \vee T_2^n))$ is isomorphic to $\underbrace{(Z \times Z \times \cdots \times Z)}_{2n-terms} * \underbrace{Z \times Z \times \cdots \times Z}_{n-terms}$. □

Theorem 12 *Let $F : M \to M$ is a folding by cut or with singularity , and M is a manifold homeomorphic to S^1 or T^1. Then, there are unfoldings $unf : F(M) \subset M \to M$ such that $\pi_1(\lim\limits_{n\to\infty} unf_n(F(M))$ is isomorphic to Z or $Z \times Z$.*

Proof We have two cases following.

Case 1. Let M be a manifold homeomorphic to S^1 , if $F : S^1 \to S^1$ is a folding by cut.

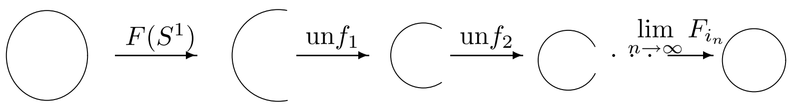

Fig.11

Then, we can define a sequence of unfoldings

$unf_1 : F(S^1) \to M_1, F(S^1) \neq S^1, M_1 \subseteq S^1,\ \ unf_2 : M_1 \to M_2, ..., unf_n : M_1 \to M_2,$

$\lim_{n\to\infty} unf_n(F(M)) = S^1$ as in Fig.11. Thus $\pi_1(\lim_{n\to\infty} unf_n(F(M)) \approx Z.$

Case 2. Let M be a manifold homeomorphic to T^1 , if $F : T^1 \to T^1$ is a folding such that $F(S_1^1) = S_1^1$ and $F(S_2^1) \neq S_2^1$. So we can define a sequence of unfoldings following.

$unf_1 : F(T^1) \to M_1,\ unf_2 : M_1 \to M_2, \cdots, unf_n : M_1 \to M_2,$

$\lim_{n\to\infty} unf_n(F(M)) = T^1$ as in Fig.12. Thus $\pi_1(\lim_{n\to\infty} unf_n(F(M)) \approx Z \times Z.$

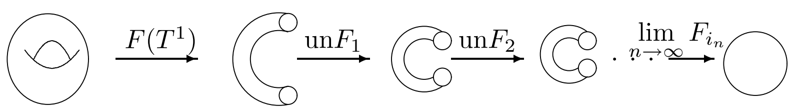

Fig.12

Therefore, $\pi_1(\lim_{n\to\infty} unf_n(F(M))$ is isomorphic to Z or $Z \times Z$. □

Corollary 4 *Let $F : M \to M$ be a folding by cut or with singularity, M is a manifold homeomorphic to S^n or $T^n, n \geq 2$. Then there are unfoldings $unf : F(M) \subset M \to M$ such that $\pi_1(\lim_{n\to\infty} unf_n(F(M))$ is the identity group or a free Abelian group of rank $2n$.*

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